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COMMENT

Lacunarity and universality

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Abstract. It is shown that the new expression for the lacunarity of Sierpinski carpets proposed by Taguchi is not appropriate and that the new expression of lacunarity, fractal dimension and connectivity together do not specify uniquely critical exponents of Ising models with spins on the carpet.

In recent years, there has been much interest in the study of phase transitions on fractals. Lacunarity is one of the geometric parameters introduced by Mandelbrot (1982) to characterise fractals. Its intended function is to measure the extent of the failure of a fractal to be translationally invariant or the degree of homogeneity of a fractal. An expression of lacunarity for Sierpinski carpets was proposed by Gefen *et al* (1984, hereafter referred to as GAM) who found that the critical exponents depend on the fractal dimension D, the connectivity Q and the lacunarity L. The GAM expression of L fails to ensure its zero value to be a necessary and sufficient condition for a translationally invariant fractal. Lin and Yang (1986) and Wu and Hu (1987) proposed other expressions of L to make it satisfy two conditions.

(i) The lacunarity $L=0$ if and only if the fractal is translationally invariant.

(ii) The lacunarity L decreases with increasing homogeneity of the fractal.

Wu and Hu (1987) and Hao and Yang (1987) studied the Ising model and Potts model on Sierpinski carpets separately and had shown that the three parameters—D, Q and L—are not sufficient to characterise the universality class. Wu and Hu pointed out that it was still a question whether or not a complete or finite set of universality criteria exists.

Taguchi (1987), however, recently proposed a new expression of lacunarity and claimed universality when the new expression of L is used, together with the fractal dimension D and the connectivity Q. In the present comment, we show that the new expression of L, the fractal dimension D and the connectivity Q cannot classify the carpets according to universality.

As an addition to the above two conditions for lacunarity to satisfy, Taguchi proposed the following third condition.

(iii) If two systems have equal lacunarity, they belong to the same universality class.

The new expression of lacunarity proposed by Taguchi is

$$L^{(m)}=\frac{1}{b-1}\sum_{s=2}^{b} L^{(m)}(s) \qquad (1)$$

where

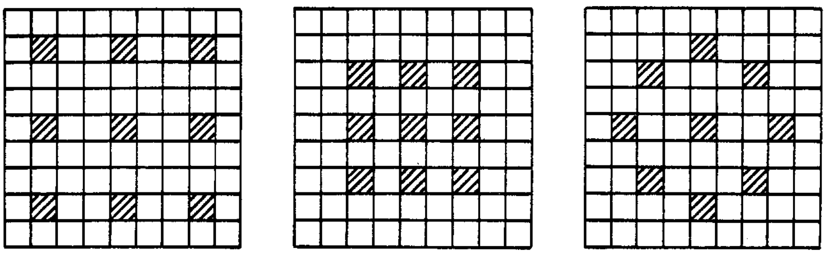
$$L^{(m)}(s)=\frac{1}{\bar{n}(s)}\left(\frac{1}{n(s)}\sum_i (n_i(s)-\bar{n}(s))^2\right)^{1/2} \tag{2}$$

$n(s)$ is the number of square subarrays of $s\times s$ cells in an array of $b^m\times b^m$ cells (it can easily be seen that $n(s)=(b^m-s+1)^2$), $n_i(s)$ represents the number of non-eliminated cells in the ith $s\times s$ covering and

$$\bar{n}(s)=\sum_i n_i(s)/n(s). \tag{3}$$

That is to say, the average of $L^{(m)}(s)$ is taken over an array of $b^m\times b^m$ cells, i.e. the m stage of the construction, instead of $b\times b$ cells as used in the GAM, Lin and Yang (1986) and Wu and Hu (1987) expressions.

The above condition (iii) is not related directly to the original concept of lacunarity (and we believe it may be contrary to condition (ii)). How can we get an expression of L that satisfies all the above three conditions? Taguchi claimed that equation (1) is such an expression. However, the values of $L^{(2)}$ of the carpets in figure 1 show that it is not the case. In figure 1, (a) is obviously more homogeneous than (b) and L_a should be less than L_b, but the new expression gave almost the same value of lacunarity for (a) and (b) (0.293 and 0.292, respectively). In figure 2 of Taguchi's (1987) paper, (c) is obviously more homogeneous than (d), but the new expression gave almost the same values of lacunarity for (c) and (d) (0.671 and 0.672, respectively). So we see this new expression does not agree with the original definition—lacunarity represents a degree of homogeneity.



(a) (b) (c)

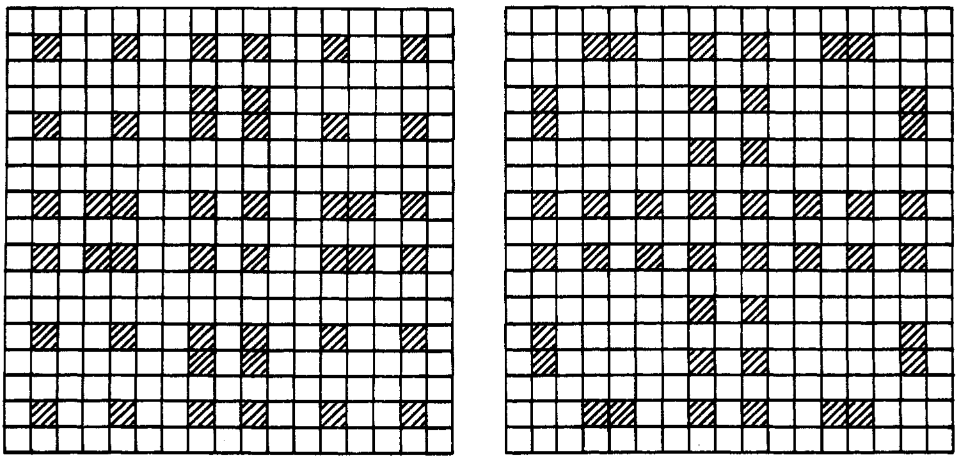
Figure 1. Three types of the first stage of the Sierpinski carpet with $b=9$, $l^2=9$, fractal dimension $D=1.946$, connectivity $Q=0.815$ and almost the same values of $L^{(2)}$. Nevertheless, the recursion relations for (c) are different from those for (a) and (b).

The reason why $L_a^{(m)}$ equals $L_b^{(m)}$ in figure 1 is that the average of $L^{(m)}$ for s is taken only from 2 to b, instead of 2 to b^m or 1 to b^m. That means the covering ($s\times s$ cells) is much less than the whole array of $b^m\times b^m$ cells which is taken over for averaging, so that $L^{(m)}$ depends mainly on the sizes and shapes of the cutout subsquares but the distribution of them, unless they combine to become some larger cutout subsquares.

The expression $L^{(m)}$ was designed such that $L_a^{(m)}=L_b^{(m)}$ in figure 1, so that the two carpets (a) and (b) will have the same D, Q and $L^{(m)}$. As we have shown, the two carpets have the same recursion relations and the same critical exponents (Wu and Hu 1987, Hao and Yang 1987). It would thus seem that D, Q and $L^{(m)}$ can classify the carpets according to universality.

If this were the fact, we might ignore the disagreement of $L^{(m)}$ with the original concept of lacunarity and take $L^{(m)}$ as a useful parameter to classify fractals. However, figure 1(c) is a counterexample. Obviously, the reason that $L_a^{(m)}$ equals $L_b^{(m)}$ in figure 1, as we have just argued, can also be used in figure 1(c). That is to say, the three carpets in figure 1 must have approximately the same values of $L^{(m)}$. (In fact, we have $L^{(2)}(9)=0.235$ for figure 1(c), which is similar to the values for figures 1(a) and (b) (0.239 and 0.233, respectively). Considering the agreement of $L_a^{(2)}(s)$ with $L_b^{(2)}(s)$ for all s from 2-9 shown in table 2 of Taguchi (1987), it is no longer necessary to compute the other $L^{(2)}(s)$ for figure 1(c).) Nevertheless, figure 1(c) has different recursion relations from those of figures 1(a) and (b). They cannot have the same critical behaviours, though they have the same values of D, Q and $L^{(m)}$. Another more convincing example is shown in figure 2. The two carpets in figure 2 have the same cutout subsquares but different distributions of them. As we have just argued, they must have the same values of $L^{(m)}$, as well as the same values of D and Q. However, figure 2(a) belongs to the carpets of the first kind and figure 2(b) the second kind in the classification of Wu and Hu (1987). (The difference between the two kinds of carpets is in whether or not there is a row or a column of the lattice which consists entirely of K_w bonds.) As Wu and Hu have shown, the two kinds of carpets have entirely different critical behaviours. (For example, the fixed point F lies on the tanh $K=1$ axis for the second kind of carpet and this is not the case for the first kind of carpet. The dependence of the critical exponents on D and Q are also different for the two kinds of carpets.) So we see, the new expression of lacunarity, $L^{(m)}$, together with the fractal dimension D and the connectivity Q, cannot classify the carpets according to universality.

As to the inadequate features of the Lin and Yang (1986) expression of lacunarity mentioned by Taguchi (1987), Wu and Hu (1987) had already discussed it and pointed out that the $L(s)$, for s less than l, should also be taken into account in the averaging of L. Furthermore, they suggested that $\bar{n}(s)$ in the Lin and Yang expression of L



(a) (b)

Figure 2. Two types of the first stage of the Sierpinski carpet with $b=17$, $l^2=44$. $D=1.942$, $Q=0.776$ and almost the same values of $L^{(2)}$. Nevertheless, the two figures belong to different kinds of carpets and have different critical behaviours.

should be replaced by $\bar{n}'(s)=s^2(b^2-l^2)/b^2$ to improve the values of $L(s)$. With the Wu and Hu expression of L ($L_3(1, b)$ in the notation of Wu and Hu (1987)), the values of lacunarity reflect correctly the homogeneity of the fractals in figure 2 of Taguchi's (1987) paper. For the four carpets of (a)-(d), the Wu and Hu expression of lacunarity gives 0.156, 0.191, 0.211 and 0.293, respectively, which is obviously much better than the results of the Lin and Yang expression and Taguchi's new expression (see Taguchi 1987).

The idea proposed by Taguchi is useful in that the average of $L(s)$ may be taken over an array of $b^m \times b^m$ cells, i.e. the m stage of the construction. But if the average of $L^{(m)}$ over s is taken from 1 to b^m, instead of 2 to b assumed by Taguchi, we may get a better measure of lacunarity. That is

$$L'^{(m)}=\frac{1}{b^m}\sum_{s=1}^{b^m} L^{(m)}(s) \tag{4}$$

where

$$L^{(m)}(s)=\frac{1}{\bar{n}'(s)}\left(\frac{1}{n(s)}\sum_i (n_i(s)-\bar{n}'(s))^2\right)^{1/2}. \tag{5}$$

Equation (5) is similar to equation (2). but $\bar{n}(s)$ is replaced by

$$\bar{n}'(s)=s^2(b^2-l^2)^m/(b^m\times b^m) \tag{6}$$

since $(b^2-l^2)^m/(b^m\times b^m)$ is the fraction of non-eliminated cells in an array of $b^m\times b^m$ cells, $\bar{n}'(s)$ provides an average measure of non-eliminated cells in an $s\times s$ covering. (For large m, $\bar{n}'(s)$ will be similar to $\bar{n}(s)$ and can be replaced by $\bar{n}(s)$, or vice versa.) As observed by Taguchi (1987), $L^{(m)}(b^m x)$ can converge with increasing value of m, so that the average of $L^{(m)}(b^m x)$ over x from $1/b^m$ to 1 (or the average of $L^{(m)}(s)$ over s from 1 to b^m), i.e. $L'^{(m)}$, can converge with increasing value of m. Nevertheless, since it is quite tedious to compute $L'^{(m)}$ for large m, as an approximation, we may set $m=1$ and come back to the Wu and Hu expression.

In conclusion, we have shown that the new expression of lacunarity proposed by Taguchi, $L^{(m)}$, together with the fractal dimension D and the connectivity Q, cannot classify the Sierpinski carpets according to universality. However, if the average range of s in the new expression changes from $(2, b)$ to $(1, b^m)$, we may get a better measure of lacunarity.

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